MTH 619 Thursday, November 15
Fall 2018

ADIC TOPOLOGIES

In class we went over a proof of a simplified version of the following result.

Proposition 1. *Let R be a commutative ring, I and ideal of R and $M \leq N$ an inclusion of R-modules.*

If N is noetherian then the I-adic topology on M coincides with the subspace topology inherited from N's I-adic topology.

Proof. We clearly have $I^n M \leq I^n N \cap M$, so it's enough to prove that for each positive integer n there is some m such that

$$I^m N \cap M \leq I^n M.$$

As we did in class, we may as well set $J = I^n$ and prove that for some m we have $J^m N \cap M \leq JM$. Again as in class I will furthermore assume that $JM = 0$ by passing to the inclusion $M/JM \leq N/JM$. Now the task reads as follows

$$JM = 0 \Rightarrow J^m N \cap M = \{0\} \text{ for some } m. \tag{1}$$

What we did in class was prove (1) in the case where J is principal, i.e. of the form (x) for some $x \in J$.

Since J is an ideal in a noetherian ring it is finitely generated, and we will proceed by induction on the number of generators for J, the base case of the induction (when J is principal) having already been taken care of.

Let

$$J = (x_1, \cdots, x_k),\ J_1 = (x_1, \cdots, x_{k-1}).$$

Now fix a positive integer d. If we show that for some m we have

$$J^m N \cap M \leq J_1^d N \cap M \tag{2}$$

then we will have achieved the induction step, since J_1 has fewer generators than J and d is arbitrary, so by induction we have

$$\exists d \text{ such that } J_1^d N \cap M = \{0\}.$$

In conclusion, we are left having to prove (2). Replacing R with R/J_1^d and N with $N/J_1^d N$, we may as well assume that $J_1^d = \{0\}$, i.e.

$$J = (x) + \text{ a nilpotent ideal } J_1, \tag{3}$$

having to show that in this case $J^m N \cap M = \{0\}$ for some m.

Now, we already know by the base case of the induction that $(x)^{m'} N \cap M$ vanishes for some m', and the conclusion follows from the fact that (3) implies

$$J^m \subseteq (x)^{m'}$$

for sufficiently large m. ■