$$R(x) = \frac{\alpha_0}{n} + \frac{Q(x)}{P'(x)},$$

where

$$Q(x) = \sum_{k=0}^{n-2} x^k \left\{ \alpha_0 a_{k+1} \frac{n-k-1}{n} + \sum_{j=1}^{n-1-k} \alpha_j a_{k+1+j} \right\}.$$

Now $R(x_i) = \alpha_0/n$ if, and only if, $Q(x_i) = 0$ $(i = 1, \cdots, n)$. But Q is of degree $n-2$, hence $Q \equiv 0$. It follows that the coefficients of P must satisfy the equations

$$\alpha_0 a_{k+1} \frac{n-k-1}{n} + \sum_{j=1}^{n-1-k} \alpha_j a_{k+1+j} = 0 \qquad (k = 0, 1, \cdots, n-2) \tag{3a}$$

together with the condition $I(P) = 0$, which is

$$\sum_{j=0}^{n} a_j \alpha_j = 0 \qquad (a_n = 1). \tag{3b}$$

These equations are identical with those obtainable from (2) by the use of Newton's identities which relate the sums of powers of the zeros of a polynomial with the coefficients. However, such an approach presupposes familiarity with these identities, and obscures the interpolatory origin of the formula. Of course, the existence of P does not guarantee the existence of Q_n since we can say (almost) nothing about the zeros x_i except in special cases.

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A NOTE ON THE ALGEBRAIC CLOSURE OF A FIELD

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In order to prove that a given field F has an algebraic closure, it is sufficient to prove the existence of an extension field K of F such that each nonconstant polynomial f over F splits into linear factors over K, for if such a field K exists, then the set of elements of K algebraic over F is an algebraic closure of F [**3**, p. 194]. To establish the existence of such a field K, Lang in his text *Algebra* [2] proceeds as follows. He first proves that if E is any field, then there is an extension field E_1 of E such that each nonconstant polynomial f over E has a root in E_1. Applying this result successively to $E_1, E_2, \cdots$, Lang obtains an ascending chain $E_1 \subset E_2 \subset \cdots$ of extension fields of E such that for each i, every nonconstant polynomial over E_i has a root in E_{i+1}. Then if $L=\bigcup_{i=1}^{\infty} E_i$, it is easy to show that each nonconstant polynomial over E splits into linear factors over L so that L contains an algebraic closure of E. (See [2] pp. 169–170.) The purpose of this note is to observe that in the above process, the field E_1 already contains an algebraic closure of E. This we do by establishing the following result:

THEOREM. *If K is a subfield of a field L and if each nonconstant polynomial with coefficients in K has a root in L, then each nonconstant polynomial with coefficients in K splits into linear factors in $L[X]$.*

Proof. It suffices to prove that for f a nonconstant irreducible monic polynomial with coefficients in K, f is a product of linear factors in $L[X]$. If K has characteristic 0, we define $p=1$; otherwise p denotes the characteristic of K. (In Bourbaki's terminology [1, p. 71], *p is the characteristic exponent of K.*) In a splitting field F of f over K, the roots of f all have the same multiplicity p^e for some nonnegative integer e: $f=\prod_{i=1}^{n}(X-\alpha_i)^{p^e}=\prod_{i=1}^{n}(X^{p^e}-\alpha_i^{p^e})$. Further, if $f=X^{np^e}+f_{n-1}X^{(n-1)p^e}+\cdots+f_1X^{p^e}+f_0$, and if $\beta_i=\alpha_i^{p^e}$ for each i, $\{\beta_1,\cdots,\beta_n\}$ is the set of roots of the irreducible, separable polynomial $g=x^n+f_{n-1}X^{n-1}+\cdots+f_0$ over K [3, p. 120]. The field $K(\beta_1,\cdots,\beta_n)$ is a finite, normal, separable extension of K. It is therefore a simple extension of K: $K(\beta_1,\cdots,\beta_n)=K(\gamma)$. If h is the minimal polynomial for γ over K, then $K(\gamma)$ is a splitting field for h over K and there is, by hypothesis, a root θ of h in L. The fields $K(\theta)$ and $K(\gamma)$ are each K-isomorphic to $K[X]/(h)$, and hence are K-isomorphic to each other. It follows that $k(\theta)$ is a splitting field of g over K. Therefore, if $p^e=1$—that is, if f is separable over K, then L contains a splitting field of f over K. Hence we have proved our theorem in case the algebraic closure of K in L is separable over K. In particular, our proof is complete if K is a perfect field.

In case K is not perfect, we let K_0 be the subfield of L consisting of those elements which are purely inseparable over K. We show that the field K_0 is perfect. Hence if $s\in K_0$ and if $s^{p^e}\in K$, then by hypothesis, $X^{p^{e+1}}-s^{p^e}$ has a root t in L. Therefore, $t^{p^{e+1}}\in K$, $t\in K_0$, and $(t^p-s)^{p^e}=0$, implying that $t^p=s$ so that K_0 is perfect [3, p. 124]. Further, if $q(x)=\sum_{i=0}^{r} q_iX^i$ is any nonconstant polynomial over K_0, then for some positive integer e, $[q(X)]^{p^e}=v(X)=\sum q_i^{p^e}X^{ip^e}\in K[X]$ so that $v(X)$ has a root α in L. We have $0=v(\alpha)=[q(\alpha)]^{p^e}$, implying that $q(\alpha)=0$. We conclude that K_0 is a perfect field with the property that each nonconstant polynomial over K_0 has a root in the extension field L of K_0. By our previous proof, each nonconstant polynomial over K_0 splits into linear factors in $L[X]$. This property holds then, in particular, for nonconstant polynomials over K, and this completes the proof of our theorem.

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